VALSCALC

Chapter 5 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 5: Applications of Integration.

5.1: Areas between Curves
5.2: Volumes with Cross Sections
5.3: Solids of Revolution
5.4: Shell Method
5.5: Work
5.6: Average Value of a Function

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

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Applications of Integration

Number of Questions—14

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points	Points Available
Short Questions		55
Question 12		15
Question 13		15
Question 14		15
TOTAL		**100**

Short Questions

1. Find the average value of $f(x) = x^3 - 6x^2$ on $[0, 2]$. *(5 pts.)*

2. An object traveling along the x-axis experiences a force given by $F(x) = x^4 e^{-x^5}$. Calculate the work done on the object as it moves from $x = 0$ to $x = 1$. *(5 pts.)*

3. Find the area of the region bounded by $y = x^2 + 6$ and $y = 9 - 2x$. *(5 pts.)*

4. Calculate the volume of the solid generated by rotating the region bounded by $y = \sqrt[3]{x}$, the x-axis, and the line $x = 2$ about the x-axis. *(5 pts.)*

5. Calculate c that satisfies the Mean Value Theorem for Integrals for $f(x) = \dfrac{1}{\sqrt{4x}}$ over $1 \leqslant x \leqslant 4$. *(5 pts.)*

6. The region enclosed by $y = \sqrt{9 - x^2}$, $x = 1$, $x = 2$, and the x-axis is the base of a solid whose cross sections perpendicular to the x-axis are rectangles of height $2x$. Calculate the solid's volume. *(5 pts.)*

7. A sample of gas under a pressure of 30 pounds per square inch occupies 2 cubic inches of volume. Assuming the gas's temperature remains constant, calculate the work done by the gas as its volume doubles. *(5 pts.)*

8. Calculate the area of the region bounded by $y = 3\sin x$ and $y = 3\cos x$ for $\frac{\pi}{4} \leqslant x \leqslant \pi$. *(5 pts.)*

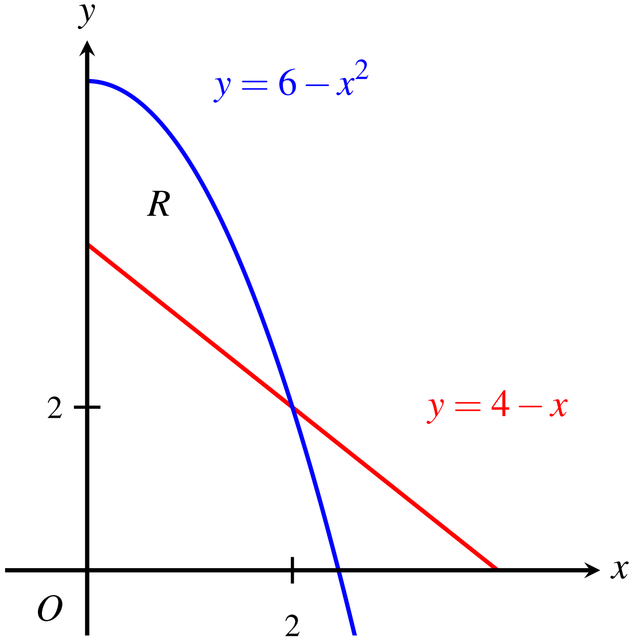
9. It takes 15 pounds of force to stretch a spring from its natural length of 1 foot to an elongated length of 1.5 feet. Calculate the work needed to pull the spring 1 foot past its natural length. *(5 pts.)*

10. A tank is 5 meters tall and has a cross-sectional area of 10 square meters. If water fills the tank up to a height of 3 meters, then how much work is needed to pump all the water to the top of the tank? (Water's specific weight is 9800 newtons per cubic meter.) *(5 pts.)*

11. A log's radius r as a function of distance x from the left end is $r(x) = 6 - x^2$ for $0 \leqslant x \leqslant 2$. *(5 pts.)*
Calculate the log's volume on this interval.

Long Questions

12. The region R is bounded by the y-axis and the graphs of $y = 6 - x^2$ and $y = 4 - x$, as shown below. Note that the graphs intersect at the point $(2, 2)$.



(a) Calculate the area of R. (2 pts.)

(b) Write, but do not solve, an integral expression that equals the volume of the solid generated by rotating R about the x-axis. (3 pts.)

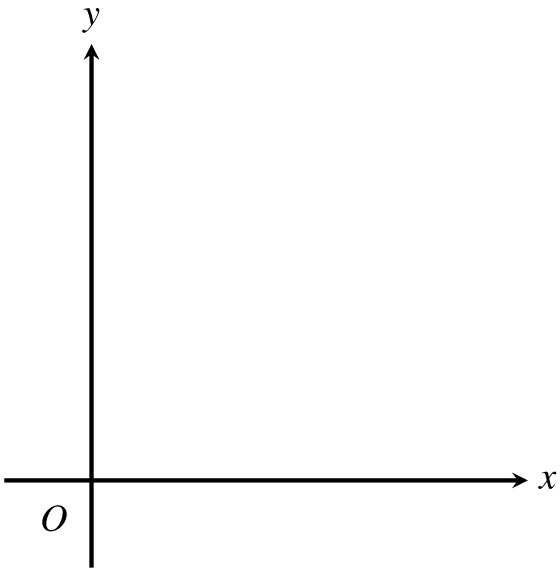
(c) Write, but do not solve, an integral expression that equals the volume of the solid generated by rotating R about the y-axis. *(3 pts.)*

(d) What is the average vertical distance from the line $y = -1$ to a point on the upper boundary of R ? *(3 pts.)*

(e) Let region R_2 be bounded by $y = 6 - x^2$, the x-axis, and the y-axis in the first quadrant. Calculate the value of k in $(0, 6)$ such that the horizontal line $y = k$ splits R_2 into two subregions of equal areas. *(4 pts.)*

13. Region S is enclosed by $y = x^2$ and $y = x$ in the first quadrant.

(a) On the following graph, sketch the two functions and shade in the region S. *(1 pt.)*

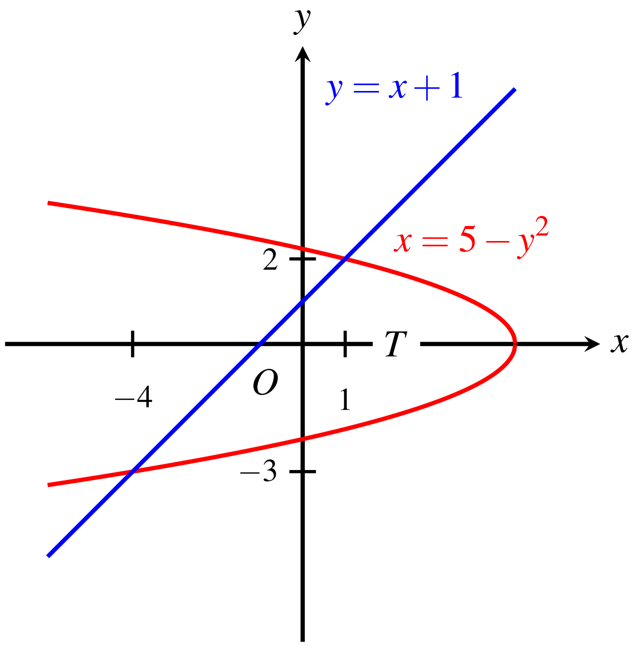


(b) Set up, but do not evaluate, an integral whose value equals the volume of the solid generated by rotating S about the line $y = -2$. *(4 pts.)*

(c) Calculate the volume of the solid generated by rotating S about the line $x = -1$. *(5 pts.)*

(d) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Calculate this solid's volume. *(5 pts.)*

14. Let T be the region bounded by the graphs of $y = x + 1$ and $x = 5 - y^2$, as in the following figure. Note that the graphs intersect at $(-4, -3)$ and $(1, 2)$.



(a) Determine the area of T. *(3 pts.)*

(b) Region T is a thin plate whose mass, in kilograms, equals twice the area found in part (a). Calculate the work needed to lift the plate 3 meters high. *(3 pts.)*

(c) Set up an integral that equals the volume of the solid generated by rotating T about the line $y = 5$. *(3 pts.)*

(d) Write, but do not evaluate, an integral that equals the volume of the solid generated by rotating T about the line $x = -6$. *(3 pts.)*

(e) Region T is the base of a solid whose cross sections perpendicular to the y-axis are semicircles. Write, but do not evaluate, an integral that equals the solid's volume. *(3 pts.)*

This marks the end of the test. The solutions and scoring rubric begin on the next page.

Short Questions (5 points each)

1. The average value is

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2-0}\int_0^2 (x^3 - 6x^2)\,\mathrm{d}x && ** \\ &= \tfrac{1}{2}\int_0^2 (x^3 - 6x^2)\,\mathrm{d}x \\ &= \tfrac{1}{2}\left(\tfrac{1}{4}x^4 - 2x^3\right)\Big|_0^2 && * \\ &= \boxed{-6} && ** \end{aligned}$$

2. The work is given by

$$W = \int_0^1 x^4 e^{-x^5}\,\mathrm{d}x\,. \qquad *$$

Substituting $u = -x^5$ gives $\mathrm{d}u = -5x^4\,\mathrm{d}x$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus, the integral becomes

$$\begin{aligned} W &= -\tfrac{1}{5}\int_0^{-1} e^u\,\mathrm{d}u && ** \\ &= -\tfrac{1}{5}e^u\Big|_0^{-1} && * \\ &= \boxed{\tfrac{1}{5}(1-e^{-1})} && * \end{aligned}$$

3. The graphs intersect when

$$\begin{aligned} x^2 + 6 &= 9 - 2x \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ \implies x &= -3\,, 1\,. && * \end{aligned}$$

Because $(9-2x) \geqslant (x^2+6)$ on $[-3,1]$, the area is

$$A = \int_{-3}^{1} \left[(9-2x)-(x^2+6)\right] \, dx \quad **$$

$$= \int_{-3}^{1} (-x^2-2x+3)\, dx$$

$$= \left(-\tfrac{1}{3}x^3 - x^2 + 3x\right)\Big|_{-3}^{1} \quad *$$

$$= \boxed{\tfrac{32}{3}} \quad *$$

4. By the Disk Method, the volume is

$$V = \pi \int_0^2 \left(\sqrt[3]{x}\right)^2 \, dx \quad **$$

$$= \pi \int_0^2 x^{2/3}\, dx$$

$$= \tfrac{3}{5}\pi x^{5/3}\Big|_0^2 \quad *$$

$$= \boxed{\tfrac{6}{5}\pi\sqrt[3]{4}} \quad **$$

5. The average value of $f(x) = \dfrac{1}{\sqrt{4x}} = \dfrac{1}{2\sqrt{x}}$ on $[1,4]$ is

$$f_{\text{avg}} = \frac{1}{4-1}\int_1^4 \frac{1}{2\sqrt{x}}\, dx \quad *$$

$$= \tfrac{1}{6}(2\sqrt{x})\Big|_1^4 \quad *$$

$$= \tfrac{1}{3}\,. \quad *$$

By the Mean Value Theorem for Integrals, a value c exists in $(1,4)$ such that $f(c) = f_{\text{avg}}$. We then have

$$\frac{1}{2\sqrt{c}} = \frac{1}{3} \quad *$$

$$2\sqrt{c} = 3$$

$$\implies c = \boxed{\tfrac{9}{4}} \quad *$$

6. At any x, a cross section to the solid is a rectangle of base $y = \sqrt{9-x^2}$ and height $2x$. Accordingly, the cross section's area is

$$A(x) = 2x\sqrt{9-x^2}\,. \quad *$$

The solid has a volume given by

$$V = \int_1^2 2x\sqrt{9-x^2}\,\mathrm{d}x\,. \quad *$$

Substituting $u = 9-x^2$ gives $\mathrm{d}u = -2x\,\mathrm{d}x$. When $x=1$, $u=8$; when $x=2$, $u=5$. The integral therefore becomes

$$V = -\int_8^5 \sqrt{u}\,\mathrm{d}u \quad *$$

$$= \int_5^8 \sqrt{u}\,\mathrm{d}u$$

$$= \tfrac{2}{3}u^{3/2}\Big|_5^8 \quad *$$

$$= \boxed{\tfrac{2}{3}\left(8\sqrt{8} - 5\sqrt{5}\right)} \quad *$$

7. With pressure $P = 30$ lb/in^2 and $V = 2$ in^3, Boyle's Law gives

$$C = PV = 60\,. \quad *$$

The volume doubles from 2 in^3 to 4 in^3, so the work done (in pound-inches) is

$$
\begin{aligned}
W &= \int_2^4 \frac{C}{V}\,\mathrm{d}V & * \\
&= \int_2^4 \frac{60}{V}\,\mathrm{d}V & * \\
&= 60\ln|V|\,\Big|_2^4 & * \\
&= \boxed{60\ln 2} & *
\end{aligned}
$$

8. Because $3\sin x \geqslant 3\cos x$ on $\left[\frac{\pi}{4}, \pi\right]$, the area is

$$
\begin{aligned}
A &= \int_{\pi/4}^{\pi} (3\sin x - 3\cos x)\,\mathrm{d}x & ** \\
&= (-3\cos x - 3\sin x)\Big|_{\pi/4}^{\pi} & * \\
&= \boxed{3(\sqrt{2}+1)} & **
\end{aligned}
$$

9. When the spring is stretched from 1 ft to 1.5 ft, its stretched distance is $x = 0.5$ ft. This process takes 15 lb of force, so by Hooke's law ($F = kx$),

$$15 = k(0.5) \implies k = 30. \qquad *$$

To stretch the spring 1 ft past its natural length, the work needed is

$$\begin{aligned} W &= \int_0^1 kx\,\mathrm{d}x \\ &= \int_0^1 30x\,\mathrm{d}x && * \\ &= 15x^2\Big|_0^1 && * \\ &= \boxed{15\,\text{ft-lb}} && ** \end{aligned}$$

10. We define the y-axis to point downward with the top of the tank being $y = 0$. Consider a thin, horizontal strip of water. Its cross-sectional area is $A(y) = 10\,\text{m}^2$, and it is located a depth of $d(y) = y$ beneath the tank's top. The water spans from $y = 2$ to $y = 5$, so the work needed to pump out all the water is

$$\begin{aligned} W &= \gamma \int_2^5 d(y)A(y)\,\mathrm{d}y \\ &= 9800 \int_2^5 10y\,\mathrm{d}y && ** \\ &= 49000y^2\Big|_2^5 && * \\ &= \boxed{1.029 \times 10^6\,\text{J}} && ** \end{aligned}$$

11. At each $x \in [0,2]$, the log's cross-sectional area is

$$\begin{aligned} A(x) &= \pi[r(x)]^2 && * \\ &= \pi(6 - x^2)^2 \\ &= \pi(x^4 - 12x^2 + 36)\,. && * \end{aligned}$$

The log's volume is given by integrating $A(x)$ from $x = 0$ to $x = 2$:

$$V = \int_0^2 \pi(x^4 - 12x^2 + 36)\,\mathrm{d}x \quad *$$

$$= \pi\left(\tfrac{1}{5}x^5 - 4x^3 + 36x\right)\Big|_0^2 \quad *$$

$$= \boxed{\frac{232\pi}{5}} \quad *$$

Long Questions (15 points each)

12. (a) Region R is bounded between $x = 0$ and $x = 2$, so its area is

$$A = \int_0^2 [(6 - x^2) - (4 - x)]\,\mathrm{d}x \quad *$$

$$= \int_0^2 (-x^2 + x + 2)\,\mathrm{d}x$$

$$= \left(-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x\right)\Big|_0^2$$

$$= \boxed{\tfrac{10}{3}} \quad *$$

(b) A gap exists between the region and the x-axis, so the Washer Method is appropriate. At any x, we consider a vertical approximating rectangle in the region. The distance from the x-axis to the rectangle's farthest side (the upper side) is

$$r_{\text{out}}(x) = 6 - x^2\,. \quad *$$

Similarly, the distance from the x-axis to the rectangle's nearest side (the bottom side) is

$$r_{\text{in}}(x) = 4 - x\,. \quad *$$

Thus, the integral for the volume is

$$V = \pi \int_0^2 \left([r_{\text{out}}(x)]^2 - [r_{\text{in}}(x)]^2\right)\,\mathrm{d}x$$

$$= \boxed{\pi \int_0^2 \left[(6 - x^2)^2 - (4 - x)^2\right]\,\mathrm{d}x} \quad *$$

(c) The best strategy is to use the Shell Method. At any x, a vertical approximating rectangle has height

$$h(x) = (6 - x^2) - (4 - x) = -x^2 + x + 2\,. \quad *$$

From the y-axis (the axis of revolution), the horizontal distance to the vertical approximating rectangle is

$$r(x) = x\,. \quad *$$

Hence, the volume is

$$\begin{aligned} V &= 2\pi \int_0^2 r(x)h(x)\,\mathrm{d}x \\ &= 2\pi \int_0^2 x(-x^2+x+2)\,\mathrm{d}x \quad * \\ &= \boxed{2\pi \int_0^2 (-x^3+x^2+2x)\,\mathrm{d}x} \end{aligned}$$

(d) The upper boundary of R is given by the curve $y = 6 - x^2$. Accordingly, the vertical distance between this curve and the line $y = -1$ is

$$d(x) = (6-x^2) - (-1) = 7 - x^2\,. \quad *$$

The average value of d on $[0, 2]$ is

$$\begin{aligned} d_{\text{avg}} &= \frac{1}{2-0}\int_0^2 (7-x^2)\,\mathrm{d}x \quad * \\ &= \tfrac{1}{2}\left(7x - \tfrac{1}{3}x^3\right)\Big|_0^2 \\ &= \boxed{\tfrac{17}{3}} \quad * \end{aligned}$$

(e) The curve $y = 6 - x^2$ may be reexpressed as $x = \sqrt{6-y}$, where region R_2 is bounded between $y = 0$ and $y = 6$. Because $y = k$ splits R_2 into two subregions of equal areas, we write

$$\int_0^k \sqrt{6-y}\,\mathrm{d}y = \int_k^6 \sqrt{6-y}\,\mathrm{d}y\,. \quad *$$

Evaluating each integral and solving for k yield

$$-\frac{2}{3}(6-y)^{3/2}\Big|_0^k = -\frac{2}{3}(6-y)^{3/2}\Big|_k^6 \qquad *$$

$$(6-y)^{3/2}\Big|_0^k = (6-y)^{3/2}\Big|_k^6$$

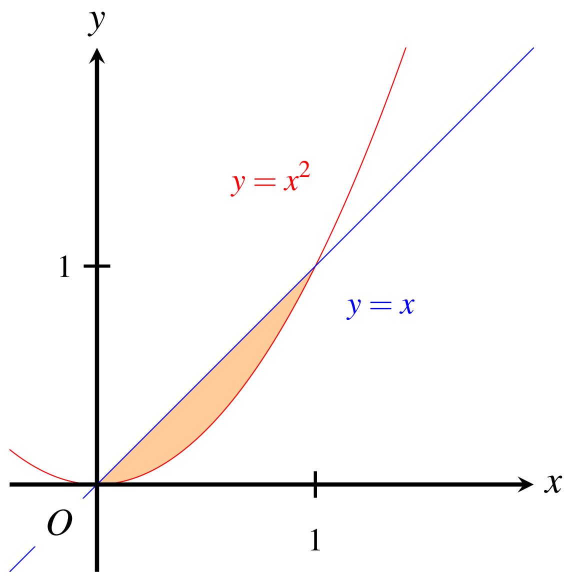
$$(6-k)^{3/2} - 6^{3/2} = 0 - (6-k)^{3/2}$$

$$2(6-k)^{3/2} = 6^{3/2}$$

$$6-k = \frac{6}{2^{2/3}}$$

$$\implies k = \boxed{6 - \frac{6}{\sqrt[3]{4}}} \qquad **$$

13. (a) The graphs are shown in the following figure. *



(b) There is a gap between region S and the line $y = -2$, so the Washer Method is appropriate. In the Washer Method, rotating about a horizontal line requires integration with x. At any x, the outer and inner radii of a washer are, respectively,

$$r_{\text{out}}(x) = x + 2\,,$$ *

$$r_{\text{in}}(x) = x^2 + 2\,.$$ *

Thus, the volume is

$$V = \pi \int_0^1 \left([r_{\text{out}}(x)]^2 - [r_{\text{in}}(x)]^2\right) \mathrm{d}x$$

$$= \boxed{\pi \int_0^1 [(x+2)^2 - (x^2+2)^2]\,\mathrm{d}x}$$ **

(c) In the Shell Method, a vertical approximating rectangle has height

$$h(x) = x - x^2$$ *

and lies a distance

$$r(x) = x + 1$$ *

away from the line $x = -1$. The volume is therefore

$$\begin{aligned} V &= 2\pi \int_0^1 r(x)h(x)\,\mathrm{d}x \\ &= 2\pi \int_0^1 (x+1)(x-x^2)\,\mathrm{d}x && * \\ &= 2\pi \int_0^1 (-x^3+x)\,\mathrm{d}x \\ &= 2\pi \left(-\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right)\Big|_0^1 && * \\ &= \boxed{\frac{\pi}{2}} && * \end{aligned}$$

(d) At any x in $[0,1]$, a square has dimensions $(x-x^2)$ by $(x-x^2)$, so its area is

$$A(x) = (x-x^2)^2. \qquad *$$

The solid's volume is therefore

$$\begin{aligned} V &= \int_0^1 A(x)\,\mathrm{d}x \\ &= \int_0^1 (x-x^2)^2\,\mathrm{d}x && * \\ &= \int_0^1 (x^4-2x^3+x^2)\,\mathrm{d}x \\ &= \left(\tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3\right)\Big|_0^1 && * \\ &= \boxed{\frac{1}{30}} && ** \end{aligned}$$

14. (a) It is easiest to integrate with y, so we reexpress $y = x+1$ as $x = y-1$. Region T is bounded between

$y=-3$ and $y=2$, so the area is

$$\begin{aligned} A &= \int_{-3}^{2} [(5-y^2)-(y-1)]\,\mathrm{d}y & * \\ &= \int_{-3}^{2} (-y^2-y+6)\,\mathrm{d}y \\ &= \left(-\tfrac{1}{3}y^3-\tfrac{1}{2}y^2+6y\right)\Big|_{-3}^{2} & * \\ &= \boxed{\tfrac{125}{6}} & * \end{aligned}$$

(b) The acceleration due to gravity is $g=9.8\,\mathrm{m/sec}^2$. To lift the plate a height h, an upward force equal in magnitude to the force of gravity (mg) must be applied over the distance h. With a mass of $m=125/3\,\mathrm{kg}$, the work done to lift the plate 3 m high is

$$\begin{aligned} W &= mgh & * \\ &= \left(\tfrac{125}{3}\right)(9.8)(3) \\ &= \boxed{1225\,\mathrm{J}} & ** \end{aligned}$$

(c) With the Shell Method, a horizontal approximating rectangle has width

$$h(y)=(5-y^2)-(y-1)=-y^2-y+6\,. \quad *$$

From the line $y=5$, the vertical distance to the approximating rectangle is

$$r(y)=5-y\,. \quad *$$

Hence, the volume is

$$\begin{aligned} V &= 2\pi\int_{-3}^{2} r(y)h(y)\,\mathrm{d}y \\ &= \boxed{2\pi\int_{-3}^{2}(5-y)(-y^2-y+6)\,\mathrm{d}y} & * \end{aligned}$$

(d) By using the Washer Method, we consider an approximating horizontal rectangle in the region T.

The distance between $x=-6$ and the rectangle's farther side is

$$r_{\text{out}}(y) = (5-y^2)+6 = 11-y^2\,. \qquad *$$

Likewise, the distance between $x=-6$ and the rectangle's nearer side is

$$r_{\text{in}}(y) = (y-1)+6 = y+5\,. \qquad *$$

Therefore, the volume of the solid of revolution is

$$\begin{aligned} V &= \pi\int_{-3}^{2}\left([r_{\text{out}}(y)]^2-[r_{\text{in}}(y)]^2\right)\,\mathrm{d}y \\ &= \boxed{\pi\int_{-3}^{2}\left[(11-y^2)^2-(y+5)^2\right]\,\mathrm{d}y} \qquad * \end{aligned}$$

(e) At any $y\in[-3,2]$, a semicircle has radius

$$r(y) = \frac{(5-y^2)-(y-1)}{2} = \frac{-y^2-y+6}{2}\,, \qquad *$$

so its area is

$$A(y) = \tfrac{1}{2}\pi[r(y)]^2 = \frac{\pi}{8}(-y^2-y+6)^2\,. \qquad *$$

Consequently, the volume of the solid is

$$\begin{aligned} V &= \int_{-3}^{2} A(y)\,\mathrm{d}y \\ &= \boxed{\frac{\pi}{8}\int_{-3}^{2}(-y^2-y+6)^2\,\mathrm{d}y} \qquad * \end{aligned}$$